QUADRATIC REVERSES OF THE TRIANGLE INEQUALITY IN INNER PRODUCT SPACES

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Abstract. Some sharp quadratic reverses for the generalised triangle inequality in inner product spaces and applications are given.

1. Introduction

In 1966, J.B. Diaz and F.T. Metcalf [1] proved the following reverse of the triangle inequality in the general settings of inner product spaces:

Theorem 1. *Let a be a unit vector in the inner product space $(H;\langle\cdot,\cdot\rangle)$ over the real or complex number field $\mathbb{K}$. Suppose that the vectors $x_i \in H\setminus\{0\}$, $i\in\{1,\dots,n\}$ satisfy*

$$0 \leq r \leq \frac{\operatorname{Re}\langle x_i, a\rangle}{\|x_i\|}, \qquad i \in \{1,\dots,n\}. \tag{1.1}$$

Then

$$r\sum_{i=1}^{n}\|x_i\| \leq \left\|\sum_{i=1}^{n} x_i\right\|, \tag{1.2}$$

where equality holds if and only if

$$\sum_{i=1}^{n} x_i = r\left(\sum_{i=1}^{n}\|x_i\|\right)a. \tag{1.3}$$

For some similar results valid for semi-inner products in normed spaces, see [3] and [4].

In the same spirit, but providing a somewhat simpler sufficient condition with a clear geometrical meaning, we note the following result obtained by the author in [2]:

Theorem 2. *Let a be as above and $\rho\in(0,1)$. If $x_i \in H$, $i\in\{1,\dots,n\}$ are such that*

$$\|x_i - a\| \leq \rho \quad \text{for each} \quad i \in \{1,\dots,n\}, \tag{1.4}$$

then we have the inequality

$$\sqrt{1-\rho^2}\sum_{i=1}^{n}\|x_i\| \leq \left\|\sum_{i=1}^{n} x_i\right\|, \tag{1.5}$$

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with equality if and only if

$$\sum_{i=1}^{n} x_i = \sqrt{1-\rho^2}\left(\sum_{i=1}^{n}\|x_i\|\right)a. \tag{1.6}$$

In a complementary direction providing reverses of the triangle inequality in its additive form, i.e., upper bounds for the nonnegative difference

$$\sum_{i=1}^{n}\|x_i\| - \left\|\sum_{i=1}^{n} x_i\right\|,$$

we note the following recent result obtained in [2]:

Theorem 3. *Let a be as above and $x_i \in H$, $k_i \geq 0$, $i \in \{1,\dots,n\}$ such that*

$$\|x_i\| - \operatorname{Re}\langle a, x_i\rangle \leq k_i \quad \text{for each} \quad i \in \{1,\dots,n\}, \tag{1.7}$$

then we have the inequality

$$0 \leq \sum_{i=1}^{n}\|x_i\| - \left\|\sum_{i=1}^{n} x_i\right\| \leq \sum_{i=1}^{n} k_i. \tag{1.8}$$

The equality holds in (1.8) if and only if

$$\sum_{i=1}^{n}\|x_i\| \geq \sum_{i=1}^{n} k_i \tag{1.9}$$

and

$$\sum_{i=1}^{n} x_i = \left(\sum_{i=1}^{n}\|x_i\| - \sum_{i=1}^{n} k_i\right)a. \tag{1.10}$$

Another similar result but with a simpler condition, is the following one [2].

Theorem 4. *Let a and x_i, $i \in \{1,\dots,n\}$ be as above. If $r_i > 0$, $i \in \{1,\dots,n\}$ are such that*

$$\|x_i - a\| \leq r_i \quad \text{for each} \quad i \in \{1,\dots,n\}, \tag{1.11}$$

then we have the inequality

$$0 \leq \sum_{i=1}^{n}\|x_i\| - \left\|\sum_{i=1}^{n} x_i\right\| \leq \frac{1}{2}\sum_{i=1}^{n} r_i^2. \tag{1.12}$$

The equality holds in (1.12) if and only if

$$\sum_{i=1}^{n}\|x_i\| \geq \frac{1}{2}\sum_{i=1}^{n} r_i^2 \tag{1.13}$$

and

$$\sum_{i=1}^{n} x_i = \left(\sum_{i=1}^{n}\|x_i\| - \frac{1}{2}\sum_{i=1}^{n} r_i^2\right)a. \tag{1.14}$$

For other inequalities related to the triangle inequality, see Chapter XVII of the book [5].

The main aim of the present paper is to point out some quadratic reverses for the generalised triangle inequality, namely, sharp upper bounds for the nonnegative differences

$$\left(\sum_{i=1}^{n}\|x_i\|\right)^2-\left\|\sum_{i=1}^{n}x_i\right\|^2,$$

under various assumptions for the vectors $x_i \in H$, $i \in \{1,\dots,n\}$ involved. Some related results are established. Applications for vector-valued integrals in Hilbert spaces are also given.

2. Quadratic Reverses of the Triangle Inequality

The following lemma holds:

Lemma 1. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $x_i \in H$, $i \in \{1,\dots,n\}$ and $k_{ij} > 0$ for $1 \le i < j \le n$ such that*

$$0 \le \|x_i\|\,\|x_j\| - \operatorname{Re}\langle x_i, x_j\rangle \le k_{ij} \tag{2.1}$$

for $1 \le i < j \le n$. Then we have the following quadratic reverse of the triangle inequality

$$\left(\sum_{i=1}^{n}\|x_i\|\right)^2 \le \left\|\sum_{i=1}^{n}x_i\right\|^2 + 2\sum_{1\le i<j\le n}k_{ij}. \tag{2.2}$$

The case of equality holds in (2.2) if and only if it holds in (2.1) for each i, j with $1 \le i < j \le n$.

Proof. We observe that the following identity holds:

$$\begin{aligned}
&\left(\sum_{i=1}^{n}\|x_i\|\right)^2-\left\|\sum_{i=1}^{n}x_i\right\|^2 \\
&= \sum_{i,j=1}^{n}\|x_i\|\,\|x_j\| - \left\langle \sum_{i=1}^{n}x_i, \sum_{j=1}^{n}x_j\right\rangle \\
&= \sum_{i,j=1}^{n}\|x_i\|\,\|x_j\| - \sum_{i,j=1}^{n}\operatorname{Re}\langle x_i, x_j\rangle \\
&= \sum_{i,j=1}^{n}\left[\|x_i\|\,\|x_j\| - \operatorname{Re}\langle x_i, x_j\rangle\right] \\
&= \sum_{1\le i<j\le n}\left[\|x_i\|\,\|x_j\| - \operatorname{Re}\langle x_i, x_j\rangle\right] + \sum_{1\le j<i\le n}\left[\|x_i\|\,\|x_j\| - \operatorname{Re}\langle x_i, x_j\rangle\right] \\
&= 2\sum_{1\le i<j\le n}\left[\|x_i\|\,\|x_j\| - \operatorname{Re}\langle x_i, x_j\rangle\right].
\end{aligned} \tag{2.3}$$

Using the condition (2.1), we deduce that

$$\sum_{1\le i<j\le n}\left[\|x_i\|\,\|x_j\| - \operatorname{Re}\langle x_i, x_j\rangle\right] \le \sum_{1\le i<j\le n}k_{ij},$$

and by (2.3), we deduce the desired inequality (2.2).

The case of equality is obvious by the identity (2.3) and we omit the details. ∎

Remark 1. *From (2.2) one may deduce the coarser inequality that might be useful in some applications:*

$$0 \leq \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| \leq \sqrt{2} \left(\sum_{1 \leq i < j \leq n} k_{ij} \right)^{\frac{1}{2}} \quad \left(\leq \sqrt{2} \sum_{1 \leq i < j \leq n} \sqrt{k_{ij}} \right).$$

Remark 2. *If the condition (2.1) is replaced with the following refinement of Schwarz's inequality:*

$$(0 \leq)\, \delta_{ij} \leq \|x_i\| \|x_j\| - \operatorname{Re} \langle x_i, x_j \rangle \ \text{ for } 1 \leq i < j \leq n, \tag{2.4}$$

then the following refinement of the quadratic generalised triangle inequality is valid:

$$\left(\sum_{i=1}^{n} \|x_i\| \right)^2 \geq \left\| \sum_{i=1}^{n} x_i \right\|^2 + 2 \sum_{1 \leq i < j \leq n} \delta_{ij} \quad \left(\geq \left\| \sum_{i=1}^{n} x_i \right\|^2 \right). \tag{2.5}$$

The equality holds in the first part of (2.5) iff the case of equality holds in (2.4) for each $1 \leq i < j \leq n$.

The following result holds.

Proposition 1. *Let $(H; \langle \cdot, \cdot \rangle)$ be as above, $x_i \in H$, $i \in \{1, \dots, n\}$ and $r > 0$ such that*

$$\|x_i - x_j\| \leq r \tag{2.6}$$

for $1 \leq i < j \leq n$. Then

$$\left(\sum_{i=1}^{n} \|x_i\| \right)^2 \leq \left\| \sum_{i=1}^{n} x_i \right\|^2 + \frac{n(n-1)}{2} r^2. \tag{2.7}$$

The case of equality holds in (2.7) if and only if

$$\|x_i\| \|x_j\| - \operatorname{Re} \langle x_i, x_j \rangle = \frac{1}{2} r^2 \tag{2.8}$$

for each i, j with $1 \leq i < j \leq n$.

Proof. The inequality (2.6) is obviously equivalent to

$$\|x_i\|^2 + \|x_j\|^2 \leq 2 \operatorname{Re} \langle x_i, x_j \rangle + r^2$$

for $1 \leq i < j \leq n$. Since

$$2 \|x_i\| \|x_j\| \leq \|x_i\|^2 + \|x_j\|^2, \quad 1 \leq i < j \leq n;$$

hence

$$\|x_i\| \|x_j\| - \operatorname{Re} \langle x_i, x_j \rangle \leq \frac{1}{2} r^2 \tag{2.9}$$

for any i, j with $1 \leq i < j \leq n$.

Applying Lemma 1 for $k_{ij} := \frac{1}{2} r^2$ and taking into account that

$$\sum_{1 \leq i < j \leq n} k_{ij} = \frac{n(n-1)}{4} r^2,$$

we deduce the desired inequality (2.7). The case of equality is also obvious by the above lemma and we omit the details. ∎

In the same spirit, and if some information about the forward difference $\Delta x_k := x_{k+1} - x_k$ $(1 \leq k \leq n-1)$ are available, then the following simple quadratic reverse of the generalised triangle inequality may be stated.

Corollary 1. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space and $x_i \in H$, $i \in \{1, \dots, n\}$. Then we have the inequality*

$$\left(\sum_{i=1}^{n} \|x_i\| \right)^2 \leq \left\| \sum_{i=1}^{n} x_i \right\|^2 + \frac{n(n-1)}{2} \sum_{k=1}^{n-1} \|\Delta x_k\| . \tag{2.10}$$

The constant $\frac{1}{2}$ is best possible in the sense that it cannot be replaced in general by a smaller quantity.

Proof. Let $1 \leq i < j \leq n$. Then, obviously,

$$\|x_j - x_i\| = \left\| \sum_{k=i}^{j-1} \Delta x_k \right\| \leq \sum_{k=i}^{j-1} \|\Delta x_k\| \leq \sum_{k=1}^{n-1} \|\Delta x_k\| .$$

Applying Proposition 1 for $r := \sum_{k=1}^{n-1} \|\Delta x_k\|$, we deduce the desired result (2.10).

To prove the sharpness of the constant $\frac{1}{2}$, assume that the inequality (2.10) holds with a constant $c > 0$, i.e.,

$$\left(\sum_{i=1}^{n} \|x_i\| \right)^2 \leq \left\| \sum_{i=1}^{n} x_i \right\|^2 + cn(n-1) \sum_{k=1}^{n-1} \|\Delta x_k\| \tag{2.11}$$

for $n \geq 2$, $x_i \in H$, $i \in \{1, \dots, n\}$.

If we choose in (2.11), $n = 2$, $x_1 = -\frac{1}{2}e$, $x_2 = \frac{1}{2}e$, $e \in H$, $\|e\| = 1$, then we get $1 \leq 2c$, giving $c \geq \frac{1}{2}$. ∎

The following result providing a reverse of the quadratic generalised triangle inequality in terms of the sup-norm of the forward differences also holds.

Proposition 2. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space and $x_i \in H$, $i \in \{1, \dots, n\}$. Then we have the inequality*

$$\left(\sum_{i=1}^{n} \|x_i\| \right)^2 \leq \left\| \sum_{i=1}^{n} x_i \right\|^2 + \frac{n^2(n^2-1)}{12} \max_{1 \leq k \leq n-1} \|\Delta x_k\|^2 . \tag{2.12}$$

The constant $\frac{1}{12}$ is best possible in (2.12).

Proof. As above, we have that

$$\|x_j - x_i\| \leq \sum_{k=i}^{j-1} \|\Delta x_k\| \leq (j-i) \max_{1 \leq k \leq n-1} \|\Delta x_k\| ,$$

for $1 \leq i < j \leq n$.

Squaring the inequality, we get

$$\|x_j\|^2 + \|x_i\|^2 \leq 2 \operatorname{Re} \langle x_i, x_j \rangle + (j-i)^2 \max_{1 \leq k \leq n-1} \|\Delta x_k\|^2$$

for any i, j with $1 \leq i < j \leq n$, and since

$$2 \|x_i\| \|x_j\| \leq \|x_j\|^2 + \|x_i\|^2 ,$$

hence

$$0 \leq \|x_i\| \|x_j\| - \mathrm{Re} \langle x_i, x_j \rangle \leq \frac{1}{2} (j-i)^2 \max_{1 \leq k \leq n-1} \|\Delta x_k\|^2 \tag{2.13}$$

for any i, j with $1 \leq i < j \leq n$.

Applying Lemma 1 for $k_{ij} := \frac{1}{2} (j-i)^2 \max_{1 \leq k \leq n-1} \|\Delta x_k\|^2$, we can state that

$$\left(\sum_{i=1}^{n} \|x_i\| \right)^2 \leq \left\| \sum_{i=1}^{n} x_i \right\|^2 + \sum_{1 \leq i < j \leq n} (j-i)^2 \max_{1 \leq k \leq n-1} \|\Delta x_k\|^2 .$$

However,

$$\begin{aligned} \sum_{1 \leq i < j \leq n} (j-i)^2 &= \frac{1}{2} \sum_{i,j=1}^{n} (j-i)^2 = n \sum_{k=1}^{n} k^2 - \left(\sum_{k=1}^{n} k \right)^2 \\ &= \frac{n^2 \left(n^2 - 1\right)}{12} \end{aligned}$$

giving the desired inequality.

To prove the sharpness of the constant, assume that (2.12) holds with a constant $D > 0$, i.e.,

$$\left(\sum_{i=1}^{n} \|x_i\| \right)^2 \leq \left\| \sum_{i=1}^{n} x_i \right\|^2 + D n^2 \left(n^2 - 1\right) \max_{1 \leq k \leq n-1} \|\Delta x_k\|^2 \tag{2.14}$$

for $n \geq 2$, $x_i \in H$, $i \in \{1, \ldots, n\}$.

If in (2.14) we choose $n = 2$, $x_1 = -\frac{1}{2} e$, $x_2 = \frac{1}{2} e$, $e \in H$, $\|e\| = 1$, then we get $1 \leq 12D$ giving $D \geq \frac{1}{12}$. ∎

The following result may be stated as well.

Proposition 3. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space and $x_i \in H$, $i \in \{1, \ldots, n\}$. Then we have the inequality:*

$$\left(\sum_{i=1}^{n} \|x_i\| \right)^2 \leq \left\| \sum_{i=1}^{n} x_i \right\|^2 + \sum_{1 \leq i < j \leq n} (j-i)^{\frac{2}{q}} \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{2}{p}}, \tag{2.15}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

The constant $E = 1$ in front of the double sum cannot generally be replaced by a smaller constant.

Proof. Using Hölder's inequality, we have

$$\begin{aligned} \|x_j - x_i\| &\leq \sum_{k=i}^{j-1} \|\Delta x_k\| \leq (j-i)^{\frac{1}{q}} \left(\sum_{k=i}^{j-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}} \\ &\leq (j-i)^{\frac{1}{q}} \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}}, \end{aligned}$$

for $1 \leq i < j \leq n$.

Squaring the previous inequality, we get

$$\|x_j\|^2 + \|x_i\|^2 \le 2\operatorname{Re}\langle x_i, x_j\rangle + (j-i)^{\frac{2}{q}} \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^p\right)^{\frac{2}{p}},$$

for $1 \le i < j \le n$.

Utilising the same argument from the proof of Proposition 2, we deduce the desired inequality (2.15).

Now assume that (2.15) holds with a constant $E > 0$, i.e.,

$$\left(\sum_{i=1}^{n} \|x_i\|\right)^2 \le \left\|\sum_{i=1}^{n} x_i\right\|^2 + E \sum_{1\le i<j\le n} (j-i)^{\frac{2}{q}} \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^p\right)^{\frac{2}{p}},$$

for $n \ge 2$ and $x_i \in H$, $i \in \{1, \dots, n\}$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

For $n = 2$, $x_1 = -\frac{1}{2}e$, $x_2 = \frac{1}{2}e$, $\|e\| = 1$, we get $1 \le E$, showing the fact that the inequality (2.15) is sharp. ∎

The particular case $p = q = 2$ is of interest.

Corollary 2. *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space and $x_i \in H$, $i \in \{1, \dots, n\}$. Then we have the inequality:*

$$\left(\sum_{i=1}^{n} \|x_i\|\right)^2 \le \left\|\sum_{i=1}^{n} x_i\right\|^2 + \frac{(n^2-1)\,n}{6} \sum_{k=1}^{n-1} \|\Delta x_k\|^2. \tag{2.16}$$

The constant $\frac{1}{6}$ is best possible in (2.16).

Proof. For $p = q = 2$, Proposition 3 provides the inequality

$$\left(\sum_{i=1}^{n} \|x_i\|\right)^2 \le \left\|\sum_{i=1}^{n} x_i\right\|^2 + \sum_{1\le i<j\le n} (j-i) \sum_{k=1}^{n-1} \|\Delta x_k\|^2,$$

and since

$$\begin{aligned}\sum_{1\le i<j\le n} (j-i) &= 1 + (1+2) + (1+2+3) + \cdots + (1+2+\cdots+n-1)\\ &= \sum_{k=1}^{n-1} (1+2+\cdots+k) = \sum_{k=1}^{n-1} \frac{k(k+1)}{2}\\ &= \frac{1}{2}\left[\frac{(n-1)\,n\,(2n-1)}{6} + \frac{n\,(n-1)}{2}\right]\\ &= \frac{n\,(n^2-1)}{6},\end{aligned}$$

hence the inequality (2.15) is proved. The best constant may be shown in the same way as above but we omit the details. ∎

Finally, we may state and prove the following different result.

Theorem 5. *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space, $y_i \in H$, $i \in \{1, \dots, n\}$ and $M \ge m > 0$ are such that either*

$$\operatorname{Re}\langle My_j - y_i, y_i - my_j\rangle \ge 0 \quad \text{for } 1 \le i < j \le n, \tag{2.17}$$

or, equivalently,

$$\left\| y_i - \frac{M+m}{2} y_j \right\| \le \frac{1}{2}(M-m)\|y_j\| \quad \text{for } 1 \le i < j \le n. \tag{2.18}$$

Then we have the inequality

$$\left(\sum_{i=1}^{n}\|y_i\|\right)^2 \le \left\|\sum_{i=1}^{n} y_i\right\|^2 + \frac{1}{2}\cdot\frac{(M-m)^2}{M+m}\sum_{k=1}^{n-1} k\|y_{k+1}\|^2. \tag{2.19}$$

The case of equality holds in (2.19) if and only if

$$\|y_i\|\|y_j\| - \operatorname{Re}\langle y_i, y_j\rangle = \frac{1}{4}\cdot\frac{(M-m)^2}{M+m}\|y_j\|^2 \tag{2.20}$$

for each i, j with $1 \le i < j \le n$.

Proof. Firstly, observe that, in an inner product space $(H; \langle\cdot,\cdot\rangle)$ and for $x, z, Z \in H$, the following statements are equivalent:

- (i) $\operatorname{Re}\langle Z - x, x - z\rangle \ge 0$
- (ii) $\left\|x - \frac{Z+z}{2}\right\| \le \frac{1}{2}\|Z - z\|$.

This shows that (2.17) and (2.18) are obviously equivalent.

Now, taking the square in (2.18), we get

$$\|y_i\|^2 + \frac{(M-m)^2}{M+m}\|y_j\|^2 \le 2\operatorname{Re}\left\langle y_i, \frac{M+m}{2} y_j\right\rangle + \frac{1}{n}(M-m)^2\|y_j\|^2$$

for $1 \le i < j \le n$, and since, obviously,

$$2\left(\frac{M+m}{2}\right)\|y_i\|\|y_j\| \le \|y_i\|^2 + \frac{(M-m)^2}{M+m}\|y_j\|^2,$$

hence

$$2\left(\frac{M+m}{2}\right)\|y_i\|\|y_j\| \le 2\operatorname{Re}\left\langle y_i, \frac{M+m}{2} y_j\right\rangle + \frac{1}{n}(M-m)^2\|y_j\|^2,$$

giving the much simpler inequality

$$\|y_i\|\|y_j\| - \operatorname{Re}\langle y_i, y_j\rangle \le \frac{1}{4}\cdot\frac{(M-m)^2}{M+m}\|y_j\|^2, \tag{2.21}$$

for $1 \le i < j \le n$.

Applying Lemma 1 for $k_{ij} := \frac{1}{4}\cdot\frac{(M-m)^2}{M+m}\|y_j\|^2$, we deduce

$$\left(\sum_{i=1}^{n}\|y_i\|\right)^2 \le \left\|\sum_{i=1}^{n} y_i\right\|^2 + \frac{1}{2}\frac{(M-m)^2}{M+m}\sum_{1\le i<j\le n}\|y_j\|^2 \tag{2.22}$$

with equality if and only if (2.21) holds for each i, j with $1 \le i < j \le n$.

Since

$$\begin{aligned}\sum_{1\le i<j\le n}\|y_j\|^2 &= \sum_{1<j\le n}\|y_j\|^2+\sum_{2<j\le n}\|y_j\|^2+\cdots+\sum_{n-1<j\le n}\|y_j\|^2\\ &= \sum_{j=2}^{n}\|y_j\|^2+\sum_{j=3}^{n}\|y_j\|^2+\cdots+\sum_{j=n-1}^{n}\|y_j\|^2+\|y_n\|^2\\ &= \sum_{j=2}^{n}(j-1)\|y_j\|^2=\sum_{k=1}^{n-1}k\|y_{k+1}\|^2,\end{aligned}$$

hence the inequality (2.19) is obtained. ∎

3. Further Quadratic Refinements of the Triangle Inequality

The following lemma is of interest in itself as well.

Lemma 2. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $x_i\in H$, $i\in\{1,\dots,n\}$ and $k\ge 1$ with the property that:*

$$\|x_i\|\,\|x_j\|\le k\operatorname{Re}\langle x_i,x_j\rangle, \tag{3.1}$$

for each i,j with $1\le i<j\le n$. Then

$$\left(\sum_{i=1}^{n}\|x_i\|\right)^2+(k-1)\sum_{i=1}^{n}\|x_i\|^2\le k\left\|\sum_{i=1}^{n}x_i\right\|^2. \tag{3.2}$$

The equality holds in (3.2) if and only if it holds in (3.1) for each i,j with $1\le i<j\le n$.

Proof. Firstly, let us observe that the following identity holds true:

$$\begin{aligned}&\left(\sum_{i=1}^{n}\|x_i\|\right)^2-k\left\|\sum_{i=1}^{n}x_i\right\|^2 \\ &=\sum_{i,j=1}^{n}\|x_i\|\,\|x_j\|-k\left\langle\sum_{i=1}^{n}x_i,\sum_{j=1}^{n}x_j\right\rangle\\ &=\sum_{i,j=1}^{n}\left[\|x_i\|\,\|x_j\|-k\operatorname{Re}\langle x_i,x_j\rangle\right]\\ &=2\sum_{1\le i<j\le n}\left[\|x_i\|\,\|x_j\|-k\operatorname{Re}\langle x_i,x_j\rangle\right]+(1-k)\sum_{i=1}^{n}\|x_i\|^2,\end{aligned} \tag{3.3}$$

since, obviously, $\operatorname{Re}\langle x_i,x_j\rangle=\operatorname{Re}\langle x_j,x_i\rangle$ for any $i,j\in\{1,\dots,n\}$.

Using the assumption (3.1), we obtain

$$\sum_{1\le i<j\le n}\left[\|x_i\|\,\|x_j\|-k\operatorname{Re}\langle x_i,x_j\rangle\right]\le 0$$

and thus, from (3.3), we deduce the desired inequality (3.2).

The case of equality is obvious by the identity (3.3) and we omit the details. ∎

Remark 3. *The inequality (3.2) provides the following reverse of the quadratic generalised triangle inequality:*

$$0 \le \left(\sum_{i=1}^{n} \|x_i\|\right)^2 - \sum_{i=1}^{n} \|x_i\|^2 \le k \left[\left\|\sum_{i=1}^{n} x_i\right\|^2 - \sum_{i=1}^{n} \|x_i\|^2\right]. \tag{3.4}$$

Remark 4. *Since $k = 1$ and $\sum_{i=1}^{n} \|x_i\|^2 \ge 0$, hence by (3.2) one may deduce the following reverse of the triangle inequality*

$$\sum_{i=1}^{n} \|x_i\| \le \sqrt{k} \left\|\sum_{i=1}^{n} x_i\right\|, \tag{3.5}$$

provided (3.1) holds true for $1 \le i < j \le n$.

The following corollary providing a better bound for $\sum_{i=1}^{n} \|x_i\|$, holds.

Corollary 3. *With the assumptions in Lemma 2, one has the inequality:*

$$\sum_{i=1}^{n} \|x_i\| \le \sqrt{\frac{nk}{n+k-1}} \left\|\sum_{i=1}^{n} x_i\right\|. \tag{3.6}$$

Proof. Using the Cauchy-Bunyakovsky-Schwarz inequality

$$n \sum_{i=1}^{n} \|x_i\|^2 \ge \left(\sum_{i=1}^{n} \|x_i\|\right)^2$$

we get

$$(k-1) \sum_{i=1}^{n} \|x_i\|^2 + \left(\sum_{i=1}^{n} \|x_i\|\right)^2 \ge \left(\frac{k-1}{n} + 1\right) \left(\sum_{i=1}^{n} \|x_i\|\right)^2. \tag{3.7}$$

Consequently, by (3.7) and (3.2) we deduce

$$k \left\|\sum_{i=1}^{n} x_i\right\|^2 \ge \frac{n+k-1}{n} \left(\sum_{i=1}^{n} \|x_i\|\right)^2$$

giving the desired inequality (3.6). ∎

The following result may be stated as well.

Theorem 6. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space and $x_i \in H \backslash \{0\}$, $i \in \{1, \dots, n\}$, $\rho \in (0,1)$, such that*

$$\left\|x_i - \frac{x_j}{\|x_j\|}\right\| \le \rho \quad \text{for } 1 \le i < j \le n. \tag{3.8}$$

Then we have the inequality

$$\sqrt{1-\rho^2} \left(\sum_{i=1}^{n} \|x_i\|\right)^2 + \left(1 - \sqrt{1-\rho^2}\right) \sum_{i=1}^{n} \|x_i\|^2 \le \left\|\sum_{i=1}^{n} x_i\right\|^2. \tag{3.9}$$

The case of equality holds in (3.9) iff

$$\|x_i\| \|x_j\| = \frac{1}{\sqrt{1-\rho^2}} \operatorname{Re} \langle x_i, x_j \rangle \tag{3.10}$$

for any $1 \le i < j \le n$.

Proof. The condition (3.1) is obviously equivalent to

$$\|x_i\|^2 + 1 - \rho^2 \leq 2\,\mathrm{Re}\left\langle x_i, \frac{x_j}{\|x_j\|}\right\rangle$$

for each $1 \leq i < j \leq n$.

Dividing by $\sqrt{1-\rho^2} > 0$, we deduce

$$\frac{\|x_i\|^2}{\sqrt{1-\rho^2}} + \sqrt{1-\rho^2} \leq \frac{2}{\sqrt{1-\rho^2}}\,\mathrm{Re}\left\langle x_i, \frac{x_j}{\|x_j\|}\right\rangle, \tag{3.11}$$

for $1 \leq i < j \leq n$.

On the other hand, by the elementary inequality

$$\frac{p}{\alpha} + q\alpha \geq 2\sqrt{pq}, \quad p, q \geq 0,\ \alpha > 0 \tag{3.12}$$

we have

$$2\|x_i\| \leq \frac{\|x_i\|^2}{\sqrt{1-\rho^2}} + \sqrt{1-\rho^2}. \tag{3.13}$$

Making use of (3.11) and (3.13), we deduce that

$$\|x_i\|\,\|x_j\| \leq \frac{1}{\sqrt{1-\rho^2}}\,\mathrm{Re}\langle x_i, x_j\rangle$$

for $1 \leq i < j \leq n$.

Now, applying Lemma 1 for $k = \frac{1}{\sqrt{1-\rho^2}}$, we deduce the desired result. ∎

Remark 5. *If we assume that* $\|x_i\| = 1$, $i \in \{1, \dots, n\}$, *satisfying the simpler condition*

$$\|x_j - x_i\| \leq \rho \quad \text{for } 1 \leq i < j \leq n, \tag{3.14}$$

then, from (3.9), we deduce the following lower bound for $\left\|\sum_{i=1}^n x_i\right\|$, *namely*

$$\left[n + n(n-1)\sqrt{1-\rho^2}\right]^{\frac{1}{2}} \leq \left\|\sum_{i=1}^n x_i\right\|. \tag{3.15}$$

The equality holds in (3.15) iff $\sqrt{1-\rho^2} = \mathrm{Re}\langle x_i, x_j\rangle$ *for* $1 \leq i < j \leq n$.

Remark 6. *Under the hypothesis of Proposition 3, we have the coarser but simpler reverse of the triangle inequality*

$$\sqrt[4]{1-\rho^2}\sum_{i=1}^n \|x_i\| \leq \left\|\sum_{i=1}^n x_i\right\|. \tag{3.16}$$

Also, applying Corollary 3 for $k = \frac{1}{\sqrt{1-\rho^2}}$, *we can state that*

$$\sum_{i=1}^n \|x_i\| \leq \sqrt{\frac{n}{n\sqrt{1-\rho^2} + 1 - \sqrt{1-\rho^2}}}\left\|\sum_{i=1}^n x_i\right\|, \tag{3.17}$$

provided $x_i \in H$ *satisfy (3.8) for* $1 \leq i < j \leq n$.

In the same manner, we can state and prove the following reverse of the quadratic generalised triangle inequality.

Theorem 7. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $x_i \in H$, $i \in \{1,\dots,n\}$ and $M \geq m > 0$ such that either*

$$\operatorname{Re}\langle Mx_j - x_i, x_i - mx_j\rangle \geq 0 \quad \text{for } 1 \leq i < j \leq n, \tag{3.18}$$

or, equivalently,

$$\left\| x_i - \frac{M+m}{2} x_j \right\| \leq \frac{1}{2}(M-m)\|x_j\| \quad \text{for } 1 \leq i < j \leq n \tag{3.19}$$

hold. Then

$$\frac{2\sqrt{mM}}{M+m}\left(\sum_{i=1}^{n}\|x_i\|\right)^2 + \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{M+m}\sum_{i=1}^{n}\|x_i\|^2 \leq \left\|\sum_{i=1}^{n} x_i\right\|^2. \tag{3.20}$$

The case of equality holds in (3.20) if and only if

$$\|x_i\|\|x_j\| = \frac{M+m}{2\sqrt{mM}}\operatorname{Re}\langle x_i, x_j\rangle \quad \text{for } 1 \leq i < j \leq n. \tag{3.21}$$

Proof. From (3.18), observe that

$$\|x_i\|^2 + Mm\|x_j\|^2 \leq (M+m)\operatorname{Re}\langle x_i, x_j\rangle, \tag{3.22}$$

for $1 \leq i < j \leq n$. Dividing (3.22) by $\sqrt{mM} > 0$, we deduce

$$\frac{\|x_i\|^2}{\sqrt{mM}} + \sqrt{mM}\|x_j\|^2 \leq \frac{M+m}{\sqrt{mM}}\operatorname{Re}\langle x_i, x_j\rangle,$$

and since, obviously

$$2\|x_i\|\|x_j\| \leq \frac{\|x_i\|^2}{\sqrt{mM}} + \sqrt{mM}\|x_j\|^2$$

hence

$$\|x_i\|\|x_j\| \leq \frac{M+m}{2\sqrt{mM}}\operatorname{Re}\langle x_i, x_j\rangle, \quad \text{for } 1 \leq i < j \leq n.$$

Applying Lemma 2 for $k = \frac{M+m}{2\sqrt{mM}} \geq 1$, we deduce the desired result. ∎

Remark 7. *We also must note that a simpler but coarser inequality that can be obtained from (3.20) is*

$$\left(\frac{2\sqrt{mM}}{M+m}\right)^{\frac{1}{2}}\sum_{i=1}^{n}\|x_i\| \leq \left\|\sum_{i=1}^{n} x_i\right\|,$$

provided (3.18) holds true.

Finally, a different result related to the generalised triangle inequality is incorporated in the following theorem.

Theorem 8. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$, $\eta > 0$ and $x_i \in H$, $i \in \{1,\dots,n\}$ with the property that*

$$\|x_j - x_i\| \leq \eta < \|x_j\| \quad \text{for each } i,j \in \{1,\dots,n\}. \tag{3.23}$$

Then we have the following reverse of the triangle inequality

$$\sum_{i=1}^{n}\sqrt{\|x_i\|^2 - \eta^2} \leq \frac{\left\|\sum_{i=1}^{n} x_i\right\|^2}{\sum_{i=1}^{n}\|x_i\|}. \tag{3.24}$$

The equality holds in (3.24) iff

$$\|x_i\|\sqrt{\|x_j\|^2-\eta^2}=\operatorname{Re}\langle x_i,x_j\rangle \quad \textit{for each} \quad i,j\in\{1,\dots,n\}. \tag{3.25}$$

Proof. From (3.23), we have

$$\|x_i\|^2-2\operatorname{Re}\langle x_i,x_j\rangle+\|x_j\|^2\le\eta^2,$$

giving

$$\|x_i\|^2+\|x_j\|^2-\eta^2\le 2\operatorname{Re}\langle x_i,x_j\rangle, \quad i,j\in\{1,\dots,n\}.$$

On the other hand,

$$2\|x_i\|\sqrt{\|x_j\|^2-\eta^2}\le\|x_i\|^2+\|x_j\|^2-\eta^2, \quad i,j\in\{1,\dots,n\}$$

and thus

$$\|x_i\|\sqrt{\|x_j\|^2-\eta^2}\le\operatorname{Re}\langle x_i,x_j\rangle, \quad i,j\in\{1,\dots,n\}.$$

Summing over $i,j\in\{1,\dots,n\}$, we deduce the desired inequality (3.24).

The case of equality is also obvious from the above, and we omit the details. ∎

4. Applications for Vector-Valued Integral Inequalities

Let $(H;\langle\cdot,\cdot\rangle)$ be a Hilbert space over the real or complex number field, $[a,b]$ a compact interval in $\mathbb{R}$ and $\eta:[a,b]\to[0,\infty)$ a Lebesgue integrable function on $[a,b]$ with the property that $\int_a^b\eta(t)\,dt=1$. If, by $L_\eta([a,b];H)$ we denote the Hilbert space of all Bochner measurable functions $f:[a,b]\to H$ with the property that $\int_a^b\eta(t)\|f(t)\|^2dt<\infty$, then the norm $\|\cdot\|_\eta$ of this space is generated by the inner product $\langle\cdot,\cdot\rangle_\eta:H\times H\to\mathbb{K}$ defined by

$$\langle f,g\rangle_\eta:=\int_a^b\eta(t)\langle f(t),g(t)\rangle\,dt.$$

The following proposition providing a reverse of the integral generalised triangle inequality may be stated.

Proposition 4. *Let $(H;\langle\cdot,\cdot\rangle)$ be a Hilbert space and $\eta:[a,b]\to[0,\infty)$ as above. If $g\in L_\eta([a,b];H)$ is so that $\int_a^b\eta(t)\|g(t)\|^2dt=1$ and $f_i\in L_\eta([a,b];H)$, $i\in\{1,\dots,n\}$, and $M\ge m>0$ are so that either*

$$\operatorname{Re}\langle Mf_j(t)-f_i(t),f_i(t)-mf_j(t)\rangle\ge0 \tag{4.1}$$

or, equivalently,

$$\left\|f_i(t)-\frac{m+M}{2}f_j(t)\right\|\le\frac{1}{2}(M-m)\|f_j(t)\|$$

for a.e. $t\in[a,b]$ and $1\le i<j\le n$, then we have the inequality

$$\begin{aligned}\left[\sum_{i=1}^n\left(\int_a^b\eta(t)\|f_i(t)\|^2dt\right)^{1/2}\right]^2 &\le \int_a^b\eta(t)\left\|\sum_{i=1}^nf_i(t)\right\|^2dt\\ &\quad+\frac{1}{2}\cdot\frac{(M-m)^2}{m+M}\int_a^b\eta(t)\left(\sum_{k=1}^{n-1}k\|f_{k+1}(t)\|^2\right)dt.\end{aligned} \tag{4.2}$$

The case of equality holds in (4.2) if and only if

$$\left(\int_a^b \eta(t)\|f_i(t)\|^2 dt\right)^{1/2}\left(\int_a^b \eta(t)\|f_j(t)\|^2 dt\right)^{1/2} - \int_a^b \eta(t)\operatorname{Re}\langle f_i(t), f_j(t)\rangle dt = \frac{1}{4}\cdot\frac{(M-m)^2}{m+M}\int_a^b \eta(t)\|f_j(t)\|^2 dt$$

for each i,j with $1 \le i < j \le n$.

Proof. We observe that

$$\operatorname{Re}\langle Mf_j - f_i, f_i - mf_j\rangle_\eta = \int_a^b \eta(t)\operatorname{Re}\langle Mf_j(t) - f_i(t), f_i(t) - mf_j(t)\rangle dt \ge 0$$

for any i,j with $1 \le i < j \le n$.

Applying Theorem 5 for the Hilbert space $L_\eta([a,b];H)$ and for $y_i = f_i, i \in \{1,\dots,n\}$, we deduce the desired result. ∎

Another integral inequality incorporated in the following proposition holds:

Proposition 5. *With the assumptions of Proposition 4, we have*

$$\frac{2\sqrt{mM}}{m+M}\left[\sum_{i=1}^n\left(\int_a^b \eta(t)\|f_i(t)\|^2 dt\right)^{1/2}\right]^2 + \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{m+M}\sum_{i=1}^n\int_a^b \eta(t)\|f_i(t)\|^2 dt \le \int_a^b \eta(t)\left\|\sum_{i=1}^n f_i(t)\right\|^2 dt. \tag{4.3}$$

The case of equality holds in (4.3) if and only if

$$\left(\int_a^b \eta(t)\|f_i(t)\|^2 dt\right)^{1/2}\left(\int_a^b \eta(t)\|f_j(t)\|^2 dt\right)^{1/2} = \frac{M+m}{2\sqrt{mM}}\int_a^b \eta(t)\operatorname{Re}\langle f_i(t), f_j(t)\rangle dt$$

for any i,j with $1 \le i < j \le n$.

The proof is obvious by Theorem 7 and we omit the details.

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